

Canonical brackets of a toy model for the Hodge theory without its canonical conjugate momenta

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Abstract: We consider the toy model of a rigid rotor as an example of the Hodge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism and show that the *internal* symmetries of this theory lead to the derivation of canonical brackets amongst the creation and annihilation operators of the dynamical variables where the definition of the canonical conjugate momenta is *not* required. We invoke *only* the spin-statistics theorem, normal ordering and basic concepts of continuous symmetries (and their generators) to derive the canonical brackets for the model of a one $(0 + 1)$ -dimensional (1D) rigid rotor without using the definition of the canonical conjugate momenta *anywhere*. Our present method of derivation of the basic brackets is conjectured to be true for a class of theories that provide a set of tractable physical examples for the Hodge theory.

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1 Introduction

One of the earliest methods of quantization of a classical (physical) system is the standard canonical quantization scheme where the (graded)Poisson brackets of the classical mechanics are upgraded to the (anti)commutators at the quantum level. In this theoretical set-up, we invoke primarily *three* basic ideas. First, we distinguish between the fermionic and bosonic variables by invoking the idea of spin-statistics theorem. Second, we take the help of the definition of canonical conjugate momenta to obtain the momenta corresponding to all the dynamical variables of a given classical theory and define the (graded)Poisson brackets. These brackets are then elevated to the (anti)commutators between the variables and corresponding momenta in their operator form. If the equations of the motion of the theory support the existence of creation and annihilation operators, the above canonical (anti)commutators are translated into the basic (anti)commutators amongst the creation and annihilation operators (e.g. in the problem of simple harmonic oscillator of quantum mechanics) and the quantization follows (at the algebraic level amongst the creation and annihilation operators). Finally, to make the physical sense out of some of the important quantities like Hamiltonian, conserved charges, etc., it is essential to adopt the normal ordering procedure in which the creation operators are brought to the left in all the terms that are found to be present in the above mentioned physical quantities of interest in a given theory.

One can provide physical meaning to the concepts of spin-statistics theorem and normal ordering but the definition of the canonical conjugate momenta remains mathematical in nature. In our present endeavor, we demonstrate that one can perform the canonical quantization without taking the help of the definition of canonical conjugate momenta for a *class* of theories which are models for the *Hodge theory*. The latter models are physical examples where the symmetries of the theory provide the physical realizations of the de Rham cohomological operators* of differential geometry [1-5]. To be precise, in our present investigation, we take up a toy model for a rigid rotor (which is a model for the Hodge theory [6]) to demonstrate that one can quantize this theory without taking the help of canonical conjugate momenta. In fact, we exploit the idea of symmetry principles (i.e. continuous symmetries and their generators) to obtain the canonical basic brackets which are consistent with the standard canonical method of quantization for this system at the level of creation and annihilation operators.

It is crystal clear, from the above assertion, that we shall take the help of spin-statistics theorem[†] as well as normal ordering in our present endeavor but we shall *not* use canonical conjugate momenta *anywhere*. This exercise, in some sense, provides the physical meaning to the canonical conjugate momenta in the language of symmetry principles. Thus, the main result of our present investigation is the theoretical trick, we have developed over the years

*On a compact manifold without a boundary, a set of *three* operators (d, δ, Δ) is called the de Rham cohomological operators where d (with $d^2 = 0$) is the exterior derivative, $\delta = \pm * d *$ (with $\delta^2 = 0$) is the co-exterior derivative and Δ is the Laplacian operator which obey *together* the algebra: $[\Delta, d] = [\Delta, \delta] = 0$, $d^2 = \delta^2 = 0$, $\Delta = (d + \delta)^2 = \{d, \delta\}$. In the above, the $(*)$ operator is popularly known as the Hodge duality operation on a given manifold (see, e.g. [1-5] for details) and this algebra is known as Hodge algebra where Δ behaves like the Casimir operator (but *not* in the sense of the Casimir operators of the Lie algebras).

[†]For the one $(0 + 1)$ -dimensional toy model, there is no meaning of spin. However, in our present investigation, we interpret the spin-statistics theorem in the language of the (anti)commutation relations of the dynamical variables of our theory.

[7, 8], by which, we obtain the basic brackets for the model of the rigid rotor by exploiting the symmetry principles (instead of using canonical conjugate momenta) that are consistent (and in complete agreement) with the canonical quantization scheme[‡].

In our present investigation, we have exploited *six* continuous symmetry transformations to obtain the canonical brackets that are in full agreement with the (anti)commutators obtained by using the *standard* canonical method of quantization. The key point, to be noted, is that *all* the six continuous symmetries and their generators play important roles in the derivation of *all* the possible (non-)vanishing brackets that are allowed amongst six creation and six annihilation operators that are present in the normal mode expansions (see, (18) below) of the six variables of the first order Lagrangian (2) (see below). Thus, we observe that, for the 1D rigid rotor, all the continuous symmetries *together* play very crucial role in the derivation of *all* the appropriate (anti)commutators amongst the creation and annihilation operators at the quantum level.

Our present investigation is essential on the following counts. First and foremost, it is very important for us to put our ideas of previous works [7, 8] on firmer footings by applying those ideas to some new physical systems so that we could get an alternative to the canonical method of quantization for a specific class of models that are physical examples of the Hodge theory. Our present endeavor is an attempt in that direction. Second, it is always gratifying to replace some mathematical definitions by a few physical principles. In our present investigation, we have an alternative to the definition of canonical conjugate momenta in the sense that we replace it by the symmetry principles for the quantization of our present system. Third, our method of quantization adds richness and variety in theoretical physics even though it is applied to a special class of theories that are examples of the Hodge theory. Finally, our present endeavor is a part of our first few steps towards our main goal of the proof that, for the models of the Hodge theory, the definition of canonical conjugate momentum is *not* required as far as the quantization of these models is concerned within the framework of BRST formalism[§].

The material of our present investigation is organized as follows. We discuss the continuous symmetries and derive the corresponding Noether conserved charges in our Sec. 2. In our forthcoming Sec. 3, we describe the *standard* canonical quantization of a 1D model for the rigid rotor. Sec. 4 contains the derivation of basic brackets from the ghost symmetry transformations where we do not use the definition of canonical conjugate momenta. Our Sec. 5 is devoted to the derivation of (anti)commutators from the basic symmetry principles associated with the continuous (anti-)BRST symmetry transformations. We derive the (anti)commutators by taking the help of basic concepts of (anti-)co-BRST symmetry transformations and their Noether conserved charges in Sec. 6. Our Sec. 7 contains the derivation of the same brackets from the bosonic symmetry transformations. Finally, we make some concluding remarks in Sec. 8 and point out a few future directions.

In our Appendix A, we have obtained the explicit canonical basic brackets from the *standard* canonical quantization method for the sake of precise comparison with such kind

[‡] It is obvious that we have already exploited our present idea in the quantization of 2D *free* as well as *interacting* Abelian 1-form gauge theory [7, 8]. In the latter category, we have considered the topic of QED with Dirac fields (where there is a coupling between the photon and a system of charged fermionic particles).

[§]We have also shown that the $\mathcal{N} = 2$ SUSY quantum mechanical models are also a set of examples for the Hodge theory which are not discussed within the framework of BRST approach (cf. Sec. 8 below).

of brackets derived in the main body of our text. Our Appendix B is devoted to some comments on the mode expansions that have been quoted in Eq. (18) (cf. Sec. 3) of our present endeavor.

General Notations and Convention: Throughout the whole body of our text, we denote the (anti-)BRST and (anti-)dual-BRST [i.e.(anti-) co-BRST] symmetry transformations by $s_{(a)b}$ and $s_{(a)d}$, respectively. Various forms of the Lagrangians (that respect the above symmetries) have been denoted with a subscript (B) attached to them. Furthermore, we have adopted the convention of left-derivative w.r.t. fermionic variables of our theory everywhere in our present endeavor.

2 Preliminaries: Symmetries and Charges

We begin with the (anti-)BRST invariant first order Lagrangian (see e.g. [9, 6, 10]) for the rigid rotor (with mass $m = 1$) as follows:

$$L_0 = \dot{r} p_r + \dot{\theta} p_\theta - \frac{p_\theta^2}{2r^2} - \lambda(r - a) + B(\dot{\lambda} - p_r) + \frac{1}{2} B^2 - i \dot{\bar{C}} \dot{C} + i \bar{C} C, \quad (1)$$

where (r, θ) are the polar coordinates, (p_r, p_θ) are the corresponding conjugate momenta, λ is the “gauge” variable, B is the Nakanishi-Lautrup type auxiliary variable and $(\bar{C})C$ are the fermionic ($C^2 = 0 = \bar{C}^2$, $C \bar{C} + \bar{C} C = 0$) (anti-)ghost variables. Here $\dot{\lambda} = d\lambda/dt$, $\dot{r} = dr/dt$, $\dot{\theta} = d\theta/dt$, etc., are the generalized “velocities” of the dynamical variables with respect to the evolution parameter t of our theory. The auxiliary variable B is invoked to linearize the gauge-fixing term $[-(\dot{\lambda} - p_r)^2/2]$ which contains $\dot{\lambda}$ and p_r together. There are two first-class constraints on the theory which originate from $(r - a) \approx 0$ and $d/dt(r - a) \approx 0$ (where a is the radius of the circle on which a particle of unit mass ($m = 1$) moves in the system of a rigid rotor). We can get rid of one of the auxiliary variables by using the Euler-Lagrange (EL) equations of motion (e.g. $p_\theta = r^2 \dot{\theta}$). The ensuing Lagrangian

$$L_B = \dot{r} p_r + \frac{1}{2} r^2 \dot{\theta}^2 - \lambda(r - a) + B(\dot{\lambda} - p_r) + \frac{1}{2} B^2 - i \dot{\bar{C}} \dot{C} + i \bar{C} C, \quad (2)$$

respects the following off-shell nilpotent ($s_{(a)b}^2 = 0$) continuous (anti-)BRST symmetry transformations ($s_{(a)b}$) (see e.g. [9, 10, 6] for details):

$$\begin{aligned} s_b p_r &= -C, & s_b \lambda &= \dot{C}, & s_b \bar{C} &= +iB, & s_b [r, \theta, C, B] &= 0, \\ s_{ab} p_r &= -\bar{C}, & s_{ab} \lambda &= \dot{\bar{C}}, & s_{ab} C &= -iB, & s_{ab} [r, \theta, \bar{C}, B] &= 0. \end{aligned} \quad (3)$$

It is trivial to note that the off-shell nilpotency ($s_{(a)b}^2 = 0$) and absolute anticommutativity ($s_b s_{ab} + s_{ab} s_b = 0$) properties are *true* for the above transformations $s_{(a)b}$. Under the continuous symmetry transformations (3), the Lagrangian (2) of our theory transforms to the total time derivatives as:

$$s_b L_B = \frac{d}{dt} [B \dot{C} - (r - a) C], \quad s_{ab} L_B = \frac{d}{dt} [B \dot{\bar{C}} - (r - a) \bar{C}]. \quad (4)$$

Thus, the transformations (3) are the *symmetry* transformations for the action integral ($S = \int dt L_B$). The Noether charges (that emerge from the transformations (3)) are as follows:

$$Q_b = B \dot{C} - \dot{B} C, \quad Q_{ab} = B \dot{\bar{C}} - \dot{B} \bar{C}. \quad (5)$$

The conservation of the charges (according to Noether's theorem) can be proven by exploiting the following EL equations of motion (EOM)

$$\begin{aligned} \dot{p}_r + \lambda = r \dot{\theta}^2, \quad \dot{B} + (r - a) = 0, \quad B + (\dot{\lambda} - p_r) = 0, \\ B = \dot{r} \Rightarrow B = \frac{d}{dt} (r - a), \quad \ddot{C} + C = 0, \quad \ddot{\bar{C}} + \bar{C} = 0, \end{aligned} \quad (6)$$

which emerge from the Lagrangian (2). It is clear that the physicality condition with the (anti-)BRST charges $Q_{(a)b} | phys \rangle = 0$ implies that $(r - a) | phys \rangle = 0$ and $(\dot{\lambda} - p_r) | phys \rangle = 0$. Translated in terms of the auxiliary variable B , these conditions imply that $B | phys \rangle = 0$ and $\dot{B} | phys \rangle = 0$. Using the above equations of motion (6), we observe that $(\dot{\lambda} - p_r) | phys \rangle = 0$ is equivalent to $d/dt (r - a) | phys \rangle = 0$. Physically, these conditions imply that the motion of the particle is confined to a circle of radius a (i.e. $r = a$) and it remains time-evolution invariant (i.e. $d/dt (r - a) = 0$). We note, in passing, that the above equations of motion imply that $\ddot{B} + B = 0$, $\frac{d^2}{dt^2} (\dot{\lambda} - p_r) + (\dot{\lambda} - p_r) = 0$ and $\ddot{R} + R = 0$ if we identify R with $(r - a)$ (i.e. $R = (r - a)$). With this identification, the conserved (anti-)BRST charges (5) can be re-expressed as: $Q_b = R C + \dot{R} \dot{C}$, $Q_{ab} = R \bar{C} + \dot{R} \dot{\bar{C}}$.

We observe that the Lagrangian (2) respects another set of nilpotent ($s_{(a)d}^2 = 0$) and absolutely anticommuting ($s_d s_{ad} + s_{ad} s_d = 0$) (anti-)co-BRST symmetry transformations $s_{(a)d}$. These transformations are as follows (see, e.g. [6, 10]):

$$\begin{aligned} s_d \lambda &= \bar{C}, & s_d C &= i(r - a), & s_d p_r &= \dot{\bar{C}}, & s_d [B, \bar{C}, r, \theta] &= 0, \\ s_{ad} \lambda &= C, & s_{ad} \bar{C} &= -i(r - a), & s_{ad} p_r &= \dot{C}, & s_{ad} [B, C, r, \theta] &= 0. \end{aligned} \quad (7)$$

It is elementary to check that $s_{(a)d} L_B = 0$. We note that[¶] ($s_{(a)d} (\dot{\lambda} - p_r) = 0$, $s_{(a)b} B = 0$) and the nilpotency and absolute anticommutativity of $s_{(a)d}$ are valid *off-shell* where we do not use any EL-EOM. The generators of the symmetry transformations (7) are^{||}

$$Q_d = \dot{R} \bar{C} - R \dot{\bar{C}} \equiv B \bar{C} + \dot{B} \dot{\bar{C}}, \quad Q_{ad} = \dot{R} C - R \dot{C} \equiv B C + \dot{B} \dot{C}. \quad (8)$$

We note that these charges are nilpotent (i.e. $Q_{(a)d}^2 = 0$) of order two and they are absolutely anticommuting ($Q_d Q_{ad} + Q_{ad} Q_d = 0$) in nature, namely;

$$\begin{aligned} s_d Q_d &= -i \{Q_d, Q_d\} = 0, & s_d Q_{ad} &= -i \{Q_{ad}, Q_d\} = 0, \\ s_{ad} Q_{ad} &= -i \{Q_{ad}, Q_{ad}\} = 0, & s_{ad} Q_d &= -i \{Q_d, Q_{ad}\} = 0, \end{aligned} \quad (9)$$

[¶]The total gauge-fixing term remains invariant under the (anti-)co-BRST symmetry transformations $s_{(a)d}$. This is a characteristic feature of the nilpotent (anti-)co-BRST [(anti-)dual-BRST] symmetry transformations $s_{(a)d}$ for this 1D system of Hodge theory [6]. We have adopted the notation $(s_{(a)d})$ for the infinitesimal and continuous (anti-)dual-BRST [(anti-)co-BRST] symmetry transformations from our earlier work [6, 10].

^{||}It will be noted that the Noether theorem yields the charges as $Q_d = B \bar{C} - (r - a) \dot{\bar{C}}$ and $Q_{ad} = B C - (r - a) \dot{C}$. These are re-expressed as (8) by using the EL-EOM (6).

when we use the equations of motion (6). We stress that the physicality criteria with the nilpotent and conserved (anti-)co-BRST charges $Q_{(a)d} | phys \rangle = 0$ lead to the annihilation of the physical states by the operator form of the first-class constraints of the theory (as was the case with such kind of criteria with the conserved and nilpotent (anti-)BRST charges).

The anticommutator $(\{s_b, s_d\} = -\{s_{ab}, s_{ad}\} = s_w)$ of the (anti-)BRST and (anti-)co-BRST symmetry transformations leads to the definition of a unique** bosonic symmetry (s_w) in our theory [6, 10]. The transformations of variables under this symmetry are

$$\begin{aligned} s_w p_r &= i [\dot{B} - (r - a)] \equiv i (\dot{B} - R), & s_w (r, \theta, C, \bar{C}, B) &= 0, \\ s_w \lambda &= i \left[B + \frac{d}{dt} (r - a) \right] \equiv i (B + \dot{R}), \\ s_w L_B &= i \frac{d}{dt} \left[B \frac{d}{dt} (r - a) - (r - a)^2 \right] \equiv i \frac{d}{dt} (B \dot{R} - R^2), \end{aligned} \quad (10)$$

which demonstrate that the action integral $S = \int dt L_B$ remains invariant under the bosonic transformations (s_w) . The conserved charge, corresponding to the above continuous symmetry transformations, is as follows:

$$Q_w = i (R^2 + B^2) \equiv i [B \dot{R} - R \dot{B}]. \quad (11)$$

The conservation law of this charge can be proven by using the the EOM (6).

We observe that the Lagrangian L_B remains invariant under the following ghost-scale symmetry transformations for the variables of our theory, namely;

$$C \longrightarrow e^{+1\Lambda} C, \quad \bar{C} \longrightarrow e^{-1\Lambda} \bar{C}, \quad \Phi \longrightarrow e^{0\Lambda} \Phi, \quad (\Phi = r, \theta, p_r, \lambda, B), \quad (12)$$

where Λ is a global parameter and numerals in the exponential denote the ghost number of the variables. The infinitesimal version of the above transformations is:

$$s_g C = +C, \quad s_g \bar{C} = -\bar{C}, \quad s_g \Phi = 0, \quad (\Phi = r, \theta, p_r, \lambda, B), \quad (13)$$

where we have set, for the sake of brevity, the scale parameter (present in (12)) equal to one (i.e. $\Lambda = 1$). The conserved charge corresponding to (13) is:

$$Q_g = i (\bar{C} \dot{C} - \dot{\bar{C}} C), \quad \dot{Q}_g = 0. \quad (14)$$

The above charge is also the generator of transformations (13) as

$$s_g C = +i [C, Q_g] = +C, \quad s_g \bar{C} = +i [\bar{C}, Q_g] = -\bar{C}. \quad (15)$$

Similarly, the trivial ghost-scale transformations on the variables $\phi = r, \theta, B, \lambda, p_r$ can be written as $s_g \phi = -i [\phi, Q_g] = 0$ because the variables $r, \lambda, p_r, \theta, B$ commute with the ghost variables of the charge Q_g . Thus, ultimately, we conclude that there are *six* continuous symmetries in the toy model (i.e. 1D rigid rotor) of our present example of Hodge theory [6].

**The transformations $s_w = \{s_b, s_d\}$ and $\bar{s}_w = \{s_{ad}, s_{ab}\}$ look different in the beginning but it can be checked that $s_w + \bar{s}_w = 0$ when we use the appropriate EL-EOM of our present theory.

3 Canonical Quantization: Normal Mode Expansions

We note that the second term (i.e. $r^2 \dot{\theta}^2/2$) in the Lagrangian (2) does not contribute anything as far as the symmetries of the theory are concerned. For a definite kinetic energy of the rigid rotor, this term becomes a constant and, therefore, it can be ignored. In particular, if the angular velocity (i.e. $\dot{\theta}$) is constant, the term $(r^2 \dot{\theta}^2/2)$ becomes a constant (which could be a constant number). In view of these arguments, we ignore the second term of the Lagrangian. As pointed out earlier, the constraint-line of our theory is defined by the relations $(r - a) \approx 0$ and $d/dt(r - a) \approx 0$ which are the first-class constraints on our theory. If we confine our system to evolve on this constraint-line, the equations of motion (6) would reduce to the following *simple* and nice-looking form^{††}:

$$\begin{aligned} \ddot{C} + C &= 0, & \ddot{\bar{C}} + \bar{C} &= 0, & \ddot{\lambda} + \lambda &= 0, \\ \ddot{p}_r + p_r &= 0, & \ddot{R} + R &= 0, & \ddot{B} + B &= 0. \end{aligned} \quad (16)$$

We re-emphasize that the above EL equations of motion are valid for a rigid rotor with a constant kinetic energy moving on a circle of radius $r = a$ at *all times* during its physical evolution which is described by the following Lagrangian

$$L_B \longrightarrow L_B^{(0)} = \dot{r} p_r - \lambda(r - a) + B(\dot{\lambda} - p_r) + \frac{1}{2} B^2 - i \dot{\bar{C}} \dot{C} + i \bar{C} C. \quad (17)$$

This is the Lagrangian we shall focus on for the rest of our discussions.

The above EL equations of motion (16) have their solutions in terms of the mode expansions (see e.g. [9]) where the creation and annihilation operators appear at the quantum level. These mode expansions, in their explicit forms, are as follows

$$\begin{aligned} R(t) &= \frac{1}{\sqrt{2}} [s e^{-it} + s^\dagger e^{+it}], & \lambda(t) &= \frac{1}{\sqrt{2}} [d e^{-it} + d^\dagger e^{+it}], \\ C(t) &= \frac{1}{\sqrt{2}} [c e^{-it} + c^\dagger e^{+it}], & \bar{C}(t) &= \frac{1}{\sqrt{2}} [\bar{c} e^{-it} + \bar{c}^\dagger e^{+it}], \\ p_r(t) &= \frac{1}{\sqrt{2}} [k e^{-it} + k^\dagger e^{+it}], & B(t) &= \frac{1}{\sqrt{2}} [l e^{-it} + l^\dagger e^{+it}], \end{aligned} \quad (18)$$

where the time-independent dagger and non-dagger operators are the creation and annihilation operators. It is clear, from the Lagrangian (17), that we have the following canonically conjugate momenta in our present theory, namely;

$$\Pi_{(C)} = +i \dot{\bar{C}}, \quad \Pi_{(\bar{C})} = -i \dot{C}, \quad \Pi_{(\lambda)} = B, \quad \Pi_{(R)} = p_r, \quad (19)$$

which lead to the basic canonical brackets as

$$[R, \Pi_{(R)}] = i, \quad [\lambda, B] = i, \quad \{C, \Pi_{(C)}\} = i, \quad \{\bar{C}, \Pi_{(\bar{C})}\} = i, \quad (20)$$

^{††}It should be noted that the EOM (6) yield the relationship $\frac{d^2}{dt^2}(\dot{\lambda} - p_r) + (\dot{\lambda} - p_r) = 0$ without any approximation. These equations can be re-expressed as $\ddot{\lambda} + \dot{\lambda} - (\ddot{p}_r + p_r) = 0$. One of its solutions of our interest is: $\ddot{\lambda} + \lambda = 0$ together with $\ddot{p}_r + p_r = 0$ (see, also Appendix B). These relations are also derived as EL-EOM when we ignore the second term $[(r^2 \dot{\theta}^2)/2]$ from the Lagrangian (2) of our theory (cf. Sec. 2).

and the rest of the brackets are zero. It is to be noted that the above (anti)commutators reduce to the following forms in terms of the explicit variables, namely;

$$[R(t), p_r(t)] = i, \quad [\lambda(t), B(t)] = i, \quad \{C(t), \dot{\bar{C}}(t)\} = 1, \quad \{\bar{C}(t), \dot{C}(t)\} = -1. \quad (21)$$

We shall concentrate on (21) for the rest of our central analysis and arguments. The above (anti)commutators (21) can be re-expressed in terms of the creation and annihilation operators of the mode expansions (18) as

$$[s, k^\dagger] = i \equiv [s^\dagger, k], \quad \{c, c^\dagger\} = -i, \quad \{\bar{c}, c^\dagger\} = +i, \quad [d, l^\dagger] = +i \equiv [d^\dagger, l], \quad (22)$$

and the rest of the (anti)commutators are zero. In other words, we have primarily *four* non-vanishing (anti)commutators at the *quantum* level and rest of *all* the (anti)commutators are zero (see, Appendix A below) as far as the canonical quantization scheme is concerned.

We would like to lay emphasis on the fact that we have utilized the spin-statistics theorem and the mathematical definition of the canonical conjugate momenta to derive the basic canonical (anti)commutators which quantize our system of a one $(0 + 1)$ -dimensional rigid rotor. There has *not* been any urgent need to exploit the idea of normal ordering as we have not expressed the Hamiltonian of our present theory in terms of the creation and annihilation operators. However, the latter idea is also one of the important ingredients of the standard canonical quantization scheme for a given physical system. We shall see that, in our forthcoming sections, this idea of normal ordering would play an important role in the context of the *proper* physical expressions for the Noether conserved charges of our theory.

4 Ghost Symmetries: Basic Canonical Brackets

Using the mode expansions (18), we can express the conserved charge Q_g in terms of the creation and annihilation operators as

$$Q_g = \bar{c}^\dagger c - \bar{c} c^\dagger \implies :Q_g: = \bar{c}^\dagger c + c^\dagger \bar{c}, \quad (23)$$

where we have used the idea of normal ordering to re-arrange all the creation operators to the left and annihilation operators to the right so that the above conserved charge Q_g could make some physical sense for our present theory.

We exploit now the virtues of (15) in deriving the anticommutators amongst the creation and annihilation operators of the expansion for $C(t)$ and $\bar{C}(t)$. Plugging in the expansion for $C(t)$ in (15), we obtain the following

$$\begin{aligned} \{c, \bar{c}\} &= \{c, c^\dagger\} = \{c, c\} = 0, & \{c, \bar{c}^\dagger\} &= -i, \\ \{c^\dagger, \bar{c}^\dagger\} &= \{c^\dagger, c\} = \{c^\dagger, c^\dagger\} = 0, & \{c^\dagger, \bar{c}\} &= +i. \end{aligned} \quad (24)$$

Similarly, the substitution of expansion for $\bar{C}(t)$, leads to

$$\begin{aligned} \{\bar{c}, c^\dagger\} &= \{\bar{c}, c\} = \{\bar{c}, \bar{c}\} = 0, & \{\bar{c}, c^\dagger\} &= +i, \\ \{\bar{c}^\dagger, \bar{c}\} &= \{\bar{c}^\dagger, c^\dagger\} = \{\bar{c}^\dagger, \bar{c}^\dagger\} = 0, & \{\bar{c}^\dagger, c\} &= -i, \end{aligned} \quad (25)$$

where we have compared the coefficients of the exponentials^{††} e^{-it} and e^{+it} from the l.h.s. and r.h.s. of (15). The bottom-line of this discussion is the observation that the non-vanishing brackets from (15) are $\{c, \bar{c}^\dagger\} = -i$ and $\{\bar{c}, c^\dagger\} = +i$ which are exactly same as the ones derived from the usual canonical method of quantization (cf. Sec. 3 for details).

We now concentrate on the trivial ghost-scale transformations

$$s_g \Phi = i [\Phi, Q_g] = 0, \quad \Phi = B, R, \lambda, p_r. \quad (26)$$

Using the expansions for Q_g (from (23)) and the mode expansions for λ, R, p_r, B from (18), it is evident that the relation (26) leads to the derivation of the following:

$$\begin{aligned} [l, c] &= 0, & [l, c^\dagger] &= 0, & [l, \bar{c}] &= 0, & [l, \bar{c}^\dagger] &= 0, \\ [l^\dagger, c] &= 0, & [l^\dagger, c^\dagger] &= 0, & [l^\dagger, \bar{c}] &= 0, & [l^\dagger, \bar{c}^\dagger] &= 0, \\ [s, c] &= 0, & [s, c^\dagger] &= 0, & [s, \bar{c}] &= 0, & [s, \bar{c}^\dagger] &= 0, \\ [s^\dagger, c] &= 0, & [s^\dagger, c^\dagger] &= 0, & [s^\dagger, \bar{c}] &= 0, & [s^\dagger, \bar{c}^\dagger] &= 0, \\ [d, c] &= 0, & [d, c^\dagger] &= 0, & [d, \bar{c}] &= 0, & [d, \bar{c}^\dagger] &= 0, \\ [d^\dagger, c] &= 0, & [d^\dagger, c^\dagger] &= 0, & [d^\dagger, \bar{c}] &= 0, & [d^\dagger, \bar{c}^\dagger] &= 0, \\ [k, c] &= 0, & [k, c^\dagger] &= 0, & [k, \bar{c}] &= 0, & [k, \bar{c}^\dagger] &= 0, \\ [k^\dagger, c] &= 0, & [k^\dagger, c^\dagger] &= 0, & [k^\dagger, \bar{c}] &= 0, & [k^\dagger, \bar{c}^\dagger] &= 0. \end{aligned} \quad (27)$$

Ultimately, we conclude that, we have obtained *all* the brackets that emerge from the ghost-scale transformations (13) and the non-vanishing brackets are the anticommutators $\{c, \bar{c}^\dagger\} = -i$ and $\{\bar{c}, c^\dagger\} = +i$ which are consistent with the canonical anticommutators derived in Sec. 3. We lay stress on the fact that we have *not* used the definition of the canonical conjugate momenta w.r.t. C and \bar{C} in our derivations of the non-vanishing canonical anticommutators $\{c, \bar{c}^\dagger\} = -i$ and $\{\bar{c}, c^\dagger\} = +i$. Instead, we have exploited the idea of symmetry principles where the continuous symmetries and their generators play the decisive roles. We observe that the ghost-scale symmetry *alone* does not produce the non-vanishing brackets $[s, k^\dagger] = i \equiv [s^\dagger, k]$ and $[d, l^\dagger] = i \equiv [d^\dagger, l]$. Thus, other continuous symmetries of the theory are required for the complete derivation of *all* the canonical basic brackets.

5 Nilpotent (Anti-)BRST Symmetries: Fundamental (Anti)commutators

From the expressions for the (anti-)BRST charges $Q_{(a)b}$, it is clear that these can be expressed in terms of the mode expansion (cf. (18)) as

$$: Q_b := (s^\dagger c + c^\dagger s) \equiv i (c^\dagger l - l^\dagger c), \quad : Q_{ab} := (s^\dagger \bar{c} + \bar{c}^\dagger s) \equiv i (\bar{c}^\dagger l - l^\dagger \bar{c}), \quad (28)$$

^{††}This is due to the fact that the exponentials e^{-it} and e^{+it} are linearly independent of each-other as they are the solutions of the generic EOM for the variable Ψ : $(\frac{d^2}{dt^2} + 1)\Psi = 0$ where $\Psi = C, \bar{C}$. The linear independence can be proven by showing that the Wronskian (for the above second-order differential equation) turns out to be non-zero for these solutions.

where we have used the equivalent expressions for (anti-)BRST charges as

$$Q_b = B \dot{C} - \dot{B} C \equiv \dot{R} \dot{C} + R C, \quad Q_{ab} = B \ddot{\bar{C}} - \ddot{B} \bar{C} \equiv \dot{R} \ddot{\bar{C}} + R \ddot{\bar{C}}, \quad (29)$$

and taken the normal ordering into consideration in (28). The conservation law on $Q_{(a)b}$ compels that these charges should be independent of time. In other words, we note that $\dot{Q}_{(a)b} = 0$ turns out to be true if we use $\ddot{R} + R = 0$, $\ddot{\bar{C}} + \bar{C} = 0$, $\ddot{\bar{C}} + \bar{C} = 0$, $\ddot{B} + B = 0$. The above forms of the normal ordered charges (28) are automatically conserved as the terms present in the above expressions are time-independent by their very definitions. We would like to emphasize that the Noether conserved charges emerge from the action principle where the mathematical definition of the canonical conjugate momenta does *not* play any role. Thus, in our discussions, we have not used the definition of canonical conjugate momentum.

We observe that $s_{(a)b} R = 0$ (since $s_{(a)b} r = 0$ in (3)). Thus, it is clear that $s_{(a)b} R = -i [R, Q_{(a)b}] = 0$. Taking the mode expansion for $R(t)$ from (18) and that for the $Q_{(a)b}$ from (28), we find the creation and annihilation operators s and s^\dagger commute with all the creation and annihilation operators present in (28). In other words, we have the following:

$$\begin{aligned} [s, s^\dagger] &= [s, c] = [s, c^\dagger] = [s^\dagger, c] = [s^\dagger, c^\dagger] = 0, \\ [s, l] &= [s^\dagger, l] = [s, l^\dagger] = [s^\dagger, l^\dagger] = 0, \\ [s, \bar{c}^\dagger] &= [s^\dagger, \bar{c}] = [s^\dagger, \bar{c}^\dagger] = [s, \bar{c}] = 0. \end{aligned} \quad (30)$$

Thus, we have obtained a vanishing set of commutators from $s_{(a)b} R = 0 = -i [R, Q_{(a)b}]$. Now, we concentrate on the transformations $s_b C = 0$ and $s_{ab} \bar{C} = 0$. These, finally, imply the following in terms of the (anti-)BRST charges, namely;

$$s_b C = -i \{C, Q_b\} = 0, \quad s_{ab} \bar{C} = -i \{\bar{C}, Q_{ab}\} = 0. \quad (31)$$

Using the mode expansions from (18) and exploiting the explicit expressions for $Q_{(a)b}$ (from (28)), we obtain the following independent basic brackets:

$$\begin{aligned} \{c, c^\dagger\} &= [c, l] = [c, l^\dagger] = \{c, c\} = 0, \\ \{\bar{c}, \bar{c}^\dagger\} &= [\bar{c}, l] = [\bar{c}, l^\dagger] = \{\bar{c}, \bar{c}\} = 0, \end{aligned} \quad (32)$$

where we have used $Q_b = B \dot{C} - \dot{B} C = i (c^\dagger l - l^\dagger c)$ and $Q_{ab} = B \ddot{\bar{C}} - \ddot{B} \bar{C} = i (\bar{c}^\dagger l - l^\dagger \bar{c})$ because these are the forms that can be used for the computation of $s_b \bar{C} = i B$, $s_{ab} C = -i B$. Thus, once again, we have obtained some vanishing (anti)commutators from the transformations $s_b C = 0$ and $s_{ab} \bar{C} = 0$ by exploiting the idea of symmetry generators.

Now, we set out to obtain the (non-)vanishing brackets from the relations $s_b p_r = -C$ and $s_{ab} p_r = -\bar{C}$ (that are present in (3)), as:

$$s_b p_r = -i [p_r, Q_b] = -C, \quad s_{ab} p_r = -i [p_r, Q_{ab}] = -\bar{C}. \quad (33)$$

Using the expansions from (18) and expressions (28), we obtain

$$\begin{aligned} [s, k^\dagger] &= i = [s^\dagger, k], \quad [k, s] = [k^\dagger, s^\dagger] = 0, \\ [k, c] &= [k, \bar{c}] = [k, c^\dagger] = [k, \bar{c}^\dagger] = 0, \\ [k^\dagger, c] &= [k^\dagger, \bar{c}] = [k^\dagger, \bar{c}^\dagger] = [k^\dagger, c^\dagger] = 0, \end{aligned} \quad (34)$$

which shows that the non-vanishing (and consistent with the canonical brackets (22)) are the brackets $[s, k^\dagger] = i$ and its Hermitian conjugate $[s^\dagger, k] = i$. The rest of the brackets are zero because the momentum operator p_r commutes with (anti-)ghost operators. Similar exercise with the symmetry transformations

$$s_b \lambda = -i [\lambda, Q_b] = \dot{C}, \quad s_{ab} \lambda = -i [\lambda, Q_{ab}] = \dot{\bar{C}}, \quad (35)$$

leads to the following basic (anti)commutators at the level of creation and annihilation operators:

$$\begin{aligned} [d, l^\dagger] &= i = [d^\dagger, l], \quad [d, l] = 0 = [d^\dagger, l^\dagger], \\ [d, c] &= [d, c^\dagger] = [d, \bar{c}] = [d, \bar{c}^\dagger] = 0, \\ [d^\dagger, c] &= [d^\dagger, c^\dagger] = [d^\dagger, \bar{c}] = [d^\dagger, \bar{c}^\dagger] = 0. \end{aligned} \quad (36)$$

We note that the non-vanishing bracket $[d, l^\dagger] = i$ and its Hermitian conjugate $[d^\dagger, l] = i$ are same as the canonical brackets listed in (22). We focus on the transformations

$$s_b \bar{C} = -i \{\bar{C}, Q_b\} = i B, \quad s_{ab} C = -i \{C, Q_{ab}\} = -i B, \quad (37)$$

and perform the earlier exercise to obtain the non-vanishing anticommutators $\{\bar{c}, c^\dagger\} = i$, $\{c, \bar{c}^\dagger\} = -i$ that are consistent with the canonical brackets (22). The vanishing brackets from our present exercise are as follows:

$$\begin{aligned} [c, l] &= [c, l^\dagger] = [c^\dagger, l] = [c^\dagger, l^\dagger] = 0, \\ [\bar{c}, l] &= [\bar{c}, l^\dagger] = [\bar{c}^\dagger, l] = [\bar{c}^\dagger, l^\dagger] = 0, \\ \{c, c\} &= \{c^\dagger, c^\dagger\} = \{\bar{c}, \bar{c}\} = \{\bar{c}^\dagger, \bar{c}^\dagger\} = 0. \end{aligned} \quad (38)$$

We emphasize that the above brackets are also consistent with the canonical brackets (22). As pointed out earlier, we have to use here the forms of the conserved and nilpotent BRST and anti-BRST charges as: $Q_b = B \dot{C} - \dot{B} C = i (c^\dagger l - l^\dagger c)$ and $Q_{ab} = B \dot{\bar{C}} - \dot{B} \bar{C} = i (\bar{c}^\dagger l - l^\dagger \bar{c})$. We concentrate on the trivial transformations $s_b B = 0$ and $s_{ab} B = 0$. These lead to the derivation of the following vanishing brackets (with both the expressions for Q_b and Q_{ab} listed in (28)), namely;

$$\begin{aligned} [l, c] &= [l, c^\dagger] = [l^\dagger, c] = [l^\dagger, c^\dagger] = [l^\dagger, s] = [l^\dagger, s^\dagger] = [l, l^\dagger] = 0, \\ [l, \bar{c}] &= [l, \bar{c}^\dagger] = [l^\dagger, \bar{c}] = [l^\dagger, \bar{c}^\dagger] = [l, s] = [l, s^\dagger] = 0. \end{aligned} \quad (39)$$

We, finally, conclude that all the vanishing as well as non-vanishing canonical quantum brackets (i.e. basic (anti-)commutators) of the standard canonical quantization scheme can be derived from the virtues of symmetry principles *alone* where the mathematical definition of the canonical conjugate momenta w.r.t. all the dynamical variables is *not* required.

6 (Anti-)co-BRST Symmetries: Basic Brackets

Using the expansions of (18), we note that the (anti-)co-BRST charges $Q_{(a)d}$ (i.e. $Q_d = B \bar{C} + \dot{B} \dot{\bar{C}} \equiv \dot{R} \bar{C} - R \dot{\bar{C}}$ and $Q_{ad} = B C + \dot{B} \dot{C} \equiv \dot{R} C - R \dot{C}$) can be expressed as:

$$: Q_d : = l^\dagger \bar{c} + \bar{c}^\dagger l \equiv i (s^\dagger \bar{c} - \bar{c}^\dagger s), \quad : Q_{ad} : = l^\dagger c + c^\dagger l \equiv i (s^\dagger c - c^\dagger s), \quad (40)$$

where the process of normal ordering has been adopted. We are in a position now to proceed in the manner that has been followed in our previous section. It is trivial to note that $s_{(a)d}(R, B) = 0$, $s_d \bar{C} = 0$, $s_{ad} C = 0$. These can be expressed in terms of $Q_{(a)d}$ as

$$\begin{aligned} s_{(a)d} B &= -i [B, Q_{(a)d}] = 0, & s_{(a)d} R &= -i [R, Q_{(a)d}] = 0, \\ s_d \bar{C} &= -i \{\bar{C}, Q_d\} = 0, & s_{ad} C &= -i \{C, Q_{ad}\} = 0. \end{aligned} \quad (41)$$

The above brackets lead to the following basic (anti)commutators amongst the creation and annihilation operators of the normal mode expansions (18), namely;

$$\begin{aligned} [s, l^\dagger] &= [s, l] = [s, \bar{c}^\dagger] = [s, \bar{c}] = [s^\dagger, l^\dagger] = [s^\dagger, l] = [s^\dagger, \bar{c}^\dagger] = [s^\dagger, \bar{c}] = 0, \\ [s, s^\dagger] &= \{c, c\} = \{c, c^\dagger\} = \{c^\dagger, c^\dagger\} = \{\bar{c}, \bar{c}\} = \{\bar{c}, \bar{c}^\dagger\} = \{\bar{c}^\dagger, \bar{c}^\dagger\} = 0, \end{aligned} \quad (42)$$

where we have quoted *only* the independent canonical quantum brackets that emerge from $s_{(a)d} \phi = -i [\phi, Q_{(a)d}]_{\pm} = 0$ where (\pm) signs on the square bracket correspond to the (anti)commutator for the generic variables $\phi = R, B, C, \bar{C}$ being (fermionic) bosonic in nature for our theory under consideration.

We next focus on the derivation of basic brackets from the symmetry transformations $s_d \lambda = -i [\lambda, Q_d] = \bar{C}$ and $s_{ad} \lambda = -i [\lambda, Q_{ad}] = C$ where the conserved charges $Q_d = B \bar{C} + \dot{B} \dot{\bar{C}}$ and $Q_{ad} = B C + \dot{B} \dot{C}$ play important roles. Using the expansions from (18) and appropriate expressions for $Q_{(a)d}$ from (40), we obtain the following (non-)vanishing basic (anti)commutators amongst the creation and annihilation operators, namely;

$$\begin{aligned} [d, l^\dagger] &= i = [d^\dagger, l], & [d, c] &= [d, \bar{c}] = [d, c^\dagger] = [d, \bar{c}^\dagger] = 0, \\ [d, l] &= 0 = [d^\dagger, l^\dagger], & [d^\dagger, c] &= [d^\dagger, \bar{c}] = [d^\dagger, c^\dagger] = [d^\dagger, \bar{c}^\dagger] = 0. \end{aligned} \quad (43)$$

Thus, we note that the non-vanishing basic brackets $[d, l^\dagger] = i$ and its Hermitian conjugate $[d^\dagger, l] = i$ are consistent with the canonical brackets defined in our Sec. 3. Similar exercise for the transformations $s_d C = i R \equiv i(r - a)$ and $s_{ad} \bar{C} = -i R = -i(r - a)$ with the (anti-)co-BRST charges, written in the following manner, namely;

$$\begin{aligned} s_d C &= -i \{C, Q_d\} \equiv -i \{C, \dot{R} \bar{C} - R \dot{\bar{C}}\} = i R, \\ s_{ad} \bar{C} &= -i \{C, Q_{ad}\} \equiv -i \{\bar{C}, \dot{R} C - R \dot{C}\} = -i R, \end{aligned} \quad (44)$$

leads to the derivation of the following basic (non-)vanishing brackets:

$$\begin{aligned} \{c, \bar{c}^\dagger\} &= -i, & \{c^\dagger, \bar{c}\} &= +i, & \{\bar{c}^\dagger, c^\dagger\} &= \{c^\dagger, \bar{c}^\dagger\} = 0, \\ [c, s] &= [c, s^\dagger] = \{c, \bar{c}\} = 0, & [c^\dagger, s] &= [c^\dagger, s^\dagger] = 0, \\ [\bar{c}, s] &= [\bar{c}, s^\dagger] = \{\bar{c}, \bar{c}\} = 0, & [\bar{c}^\dagger, s] &= [\bar{c}^\dagger, s^\dagger] = 0. \end{aligned} \quad (45)$$

Thus, we observe that the symmetry transformations $s_d C = i R$ and $s_{ad} \bar{C} = -i R$ produce the non-vanishing anticommutators between the creation and annihilation operators for the (anti-)ghost variables as: $\{c, \bar{c}^\dagger\} = -i$ and $\{\bar{c}, c^\dagger\} = +i$ which are consistent with such basic anticommutators defined in the case of canonical method of quantization (cf. Sec. 3).

Finally, we concentrate on the transformations $s_d p_r = \dot{\bar{C}}$ and $s_{ad} p_r = \dot{C}$. These can be written (in terms of the (anti-)co-BRST charges $Q_{(a)d}$) as:

$$\begin{aligned} s_d p_r &= -i [p_r, Q_d] \equiv -i [p_r, \dot{R} \bar{C} - R \dot{\bar{C}}] = \dot{\bar{C}}, \\ s_{ad} p_r &= -i [p_r, Q_{ad}] \equiv -i [p_r, \dot{R} C - R \dot{C}] = \dot{C}. \end{aligned} \quad (46)$$

Plugging in the expansions from (18) and appropriate forms (i.e. $Q_d = i(s^\dagger \bar{c} - \bar{c}^\dagger s)$, $Q_{ad} = i(s^\dagger c - c^\dagger s)$) of the conserved (anti-)co-BRST charges $Q_{(a)d}$, we obtain the following fundamental (anti)commutators amongst the creation and annihilation operators:

$$\begin{aligned} [s, k^\dagger] &= i = [s^\dagger, k], \quad [k, \bar{c}] = [k, \bar{c}^\dagger] = [k, s] = [k^\dagger, \bar{c}] = 0, \\ [k, c] &= [k, c^\dagger] = [k^\dagger, c^\dagger] = [k^\dagger, c] = [k^\dagger, \bar{c}^\dagger] = [k^\dagger, s^\dagger] = 0. \end{aligned} \quad (47)$$

These (non-)vanishing (anti)commutators establish that the non-vanishing canonical brackets are $[s, k^\dagger] = i$ and $[s^\dagger, k] = i$. These are consistent with such canonical brackets derived in Sec. 3. Thus, we conclude that all the basic brackets, derived from the (anti-)co-BRST charges and their corresponding symmetries, are consistent with the canonical brackets (i.e. (anti-)commutators) defined in Sec. 3. by the standard canonical method.

7 Bosonic Symmetries: Fundamental Brackets

We devote time on the derivation of the basic canonical brackets that emerge from the symmetry transformations generated by the bosonic conserved charge $Q_w = i(R^2 + B^2)$ (cf. Eq. (11)) which can be re-expressed, using the equations of motion (6), as

$$Q_w = i[B \dot{R} - \dot{B} R] \equiv i(R^2 + \dot{R}^2) \equiv i(B^2 + \dot{B}^2). \quad (48)$$

The above expansions can be written, in terms of the mode expansion (18), as follows:

$$\begin{aligned} Q_w &= (l^\dagger s - l s^\dagger) \implies :Q_w: = (l^\dagger s - s^\dagger l), \\ Q_w &= i(s^\dagger s + s s^\dagger) \implies :Q_w: = 2i s^\dagger s, \\ Q_w &= i(l^\dagger l + l l^\dagger) \implies :Q_w: = 2i l^\dagger l, \end{aligned} \quad (49)$$

where the procedure of normal ordering has been adopted in the last forms of Q_w . These expressions would be suitably used for our computations of the basic canonical brackets from the symmetry principles where the *appropriate* normal ordered expression for Q_w would be utilized as the generator for the bosonic symmetry transformations.

We note, from the bosonic symmetry transformations (10), that *only* the transformations $s_w p_r$ and $s_w \lambda$ exist and rest of the variables of the theory do *not* transform at all. In particular, we observe that, the (anti-)ghost variables do not transform under s_w . We would also like to state a few words on the forms of the non-vanishing transformations $s_w p_r$ and $s_w \lambda$ (cf. (10)) which can be re-expressed as:

$$\begin{aligned} s_w p_r &= i(\dot{B} - R) \equiv -2i R \equiv 2i \dot{B}, \\ s_w \lambda &= i(\dot{R} + B) \equiv 2i B \equiv 2i \dot{R}, \end{aligned} \quad (50)$$

by using EOM (6). It can be checked that, the following combinations:

$$\begin{aligned} s_w^{(1)} p_r &= -2iR, & s_w^{(1)} \lambda &= 2iB, \\ s_w^{(2)} p_r &= 2i\dot{B}, & s_w^{(2)} \lambda &= 2i\dot{R}, \end{aligned} \quad (51)$$

are the symmetry transformations for the Lagrangian (17) and its corresponding action $S = \int dt L_B^{(0)}$ because we observe that the following is true, namely;

$$s_w^{(1)} L_B^{(0)} = i \frac{d}{dt} (B^2 - R^2), \quad s_w^{(2)} L_B^{(0)} = i \frac{d}{dt} (2\dot{R}B - R^2 - B^2). \quad (52)$$

Both the above bosonic symmetry transformations lead to the derivation of the conserved Noether charge as $Q_w = i(B^2 + R^2)$ which is also quoted in (11). The noteworthy point is that any other combinations of (50) are *not* found to be the symmetry of the Lagrangian $L_B^{(0)}$ and the corresponding action (i.e. $S = \int dt L_B^{(0)}$).

Now we dwell a bit on the derivation of the canonical basic brackets from the symmetry transformations (51) and the conserved charge Q_w defined in (49). These can be written as

$$\begin{aligned} s_w^{(1)} p_r &= -i[p_r, Q_w] = -2iR \\ \implies -i[p_r, 2is^\dagger s] &= \frac{-2i}{\sqrt{2}} (s e^{-it} + s^\dagger e^{+it}). \end{aligned} \quad (53)$$

The comparison of the coefficients of e^{-it} and e^{+it} from the l.h.s. and r.h.s. leads to the following (non-)vanishing basic canonical brackets:

$$[k, s] = [k^\dagger, s^\dagger] = 0, \quad [k, s^\dagger] = i = [k^\dagger, s]. \quad (54)$$

It is to be noted that, even though the transformations $s_w^{(2)}$, are *also* symmetry transformations for the action $S = \int dt L_B^{(0)}$, these transformations are *not* interesting to us. Let us now concentrate on the following bosonic symmetry transformations:

$$s_w^{(1)} \lambda = -i[\lambda, Q_w] = 2iB \implies -i[\lambda, 2il^\dagger l] = \frac{2i}{\sqrt{2}} (b e^{-it} + l^\dagger e^{+it}). \quad (55)$$

Plugging in the expansion for λ from (18) and taking the appropriate form of $Q_w = 2il^\dagger l$ from (49), we obtain the following (non-)vanishing basic brackets:

$$[d, l] = [d^\dagger, l^\dagger] = 0, \quad [d, l^\dagger] = i = [d^\dagger, l]. \quad (56)$$

Thus, we point out that we have derived the non-vanishing brackets as $[d, l^\dagger] = i = [d^\dagger, l]$ which are in full agreement with the canonical brackets derived in Sec. 3, (cf. (22)). We re-emphasize that even though $s_w^{(2)}$ exists as a symmetry of the Lagrangian $L_B^{(0)}$ and corresponding action, it is not interesting for our purpose. Thus, we conclude that there is a *unique* bosonic symmetry $s_w^{(1)} p_r = -2iR$, $s_w^{(1)} \lambda = 2iB$, $s_w^{(1)} (R, C, \bar{C}, B) = 0$ in our theory which is equivalent to the symmetry transformations (10). We are entitled to make the above assertion because the transformations (51) are equivalent and we have to make an appropriate choice of the transformations for our specific requirement.

We end this section with the remark that the trivial bosonic symmetry transformations $s_w^{(1)}(R, C, \bar{C}, B) = 0$ lead to the derivation of the following vanishing basic brackets:

$$\begin{aligned}
[s, s^\dagger] &= [s, l] = [s, l^\dagger] = [s^\dagger, l] = [s^\dagger, l^\dagger] = 0, \\
[c, s] &= [c, s^\dagger] = [c, l] = [c, l^\dagger] = [l, l^\dagger] = 0, \\
[c^\dagger, s] &= [c^\dagger, s^\dagger] = [c^\dagger, l] = [c^\dagger, l^\dagger] = 0, \\
[\bar{c}, s] &= [\bar{c}, s^\dagger] = [\bar{c}, l] = [\bar{c}, l^\dagger] = 0, \\
[\bar{c}^\dagger, s] &= [\bar{c}^\dagger, s^\dagger] = [\bar{c}^\dagger, l] = [\bar{c}^\dagger, l^\dagger] = 0,
\end{aligned} \tag{57}$$

which are in complete agreement with the canonical basic brackets (cf. App. A), derived in Sec. 3. In a nut-shell, we draw the conclusion that *all* the *six* continuous symmetries of our present theory lead to the derivation of basic canonical brackets that are in total agreement with the basic brackets derived by the standard canonical method of quantization.

8 Conclusions

In our present endeavor, we have provided an alternative to the *standard* canonical method of quantization for a specific model of the Hodge theory which is nothing but the 1D rigid rotor. We have not used the definition of canonical conjugate momenta w.r.t. the dynamical variables at any place in our approach which has led to the derivation of canonical basic brackets at the level of creation and annihilation operators of this theory. Our method of quantization depends heavily on the symmetry principles which provide an alternative to the definition of canonical conjugate momenta. However, we have taken the help of standard spin-statistics theorem in defining the (anti)commutators and utilized the concept of normal ordering to make sense out of the conserved Noether charges corresponding to the *six* continuous symmetries that are present in our theory. As pointed out earlier, we would like to stress that, for the 1D system, the spin-statistics theorem is only limited to the definitions of (anti)commutators. There is no meaning of *spin* quantum number in 1D.

We would like to pin-point some of the subtle features of our present investigation. To obtain the normal mode expansion (18) for all the relevant variables, we have made physically motivated approximations where we have ignored the term $[(1/2)(r^2 \dot{\theta}^2)]$ from the Lagrangian (2) because it does *not* contribute anything in the discussion of the continuous symmetries of our present theory. It has also been argued that, for a constant value of $\dot{\theta}$, this term becomes a constant in the case of a rigid rotor. As a consequence, we obtain the equations of motion: $\ddot{p}_r + p_r = 0$ and $\ddot{\lambda} + \lambda = 0$ which have very nice and simple normal mode expansion as illustrated in (18). We would like to add that, *even without* any approximation, we have the validity of the relationship: $\frac{d^2}{dt^2}(\dot{\lambda} - p_r) + (\dot{\lambda} - p_r) = 0$. One of the solutions of our interest (for this relationship) is $\ddot{\lambda} + \lambda = 0$ and $\ddot{p}_r + p_r = 0$. These solutions are *not* unique but are of utmost importance to us as they support the normal mode expansions given in (18) for $\lambda(t)$ and $p_r(t)$ which are very useful to us.

We have applied our idea of quantization scheme to the discussion of 2D *free* Abelian gauge theory which is a model for the Hodge theory (see, e.g. [7]). It was interesting to extend this work to the case if *interacting* $U(1)$ gauge theory (i.e. QED) where the 1-form gauge field couples to the Dirac fields [8]. It has been very gratifying to observe that our

method of quantization is true in the case of SUSY quantum mechanics where a SUSY harmonic oscillator is considered for its quantization [11]. We conjecture that our method of quantization would be valid for *all* the models for Hodge theory that would incorporate gauge theories, 1D toy models and SUSY theories. Having applied this method in the context of gauge theories and SUSY theories, it was a challenging problem for us to apply it to a 1D toy model. We have accomplished this goal in our present investigation for the case of a 1D rigid rotor which happens to be a toy model for the Hodge theory [6].

Our method of quantization is valid *only* for a specific class of theories which are the models for the Hodge theory. For instance, such theories are Abelian p -form ($p = 1, 2, 3$) gauge theories which have been shown to be the field theoretic models for the Hodge theory in $D = 2p$ dimensions of spacetime (see, e.g. [12-16]). These theories respect *six* continuous symmetries that lead to the derivation of canonical basic brackets amongst the creation and annihilation operators. The (non-)vanishing brackets are exactly same as the ones derived by the *standard* method of canonical quantization scheme. Of course, our method is algebraically more involved but it has aesthetic appeal in the sense that it is the symmetry principles that replace the definition of the canonical conjugate momenta. It is worth pointing out that, in a recent paper [11], we have applied our method of quantization to the supersymmetric (SUSY) $\mathcal{N} = 2$ harmonic oscillator and obtained the basic brackets from the symmetry principles. In this case, there are only *three* continuous symmetries and they lead to the derivation of precise (anti)commutators that are also obtained by the standard canonical method.

We have proposed many models for the Hodge theory which are from the domains of p -form ($p = 1, 2, 3$) gauge theories [12-16] and $\mathcal{N} = 2$ SUSY quantum mechanics [17-19]. One of the decisive features of the models for the Hodge theories, connected with the p -form gauge theories, is that these theories are always endowed with *six* continuous symmetries within the framework of BRST formalism. On the contrary, all the models of $\mathcal{N} = 2$ SUSY quantum mechanics (that have been shown to be the physical examples of Hodge theory [17-19]) respect only *three* continuous symmetries. We have established in [11] that these *three* symmetries are good enough to yield the proper (anti)commutators which are found to be exactly same as the *ones* derived by the *standard* canonical quantization method. It would be worthwhile to point out that, for this purpose, the well-known one $(0 + 1)$ -dimensional (1D) model of SUSY harmonic oscillator has been taken into consideration. There is yet another SUSY quantum mechanical model which has been shown to be the physical example for the Hodge theory [20] where, once again, only *three* continuous symmetries exist. This is the simple toy model of $\mathcal{N} = 2$ SUSY free particle. We plan to discuss its standard canonical quantization and wish to compare it with the quantization through symmetry principles. It would be nice future endeavour for us to obtain the quantization of the above models [17-19] by using our proposed *novel* method so that this idea could be firmly established [21].

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Appendix A: Brackets from the Canonical Method

We list here *all* the basic brackets (i.e. (anti)commutators) that emerge from the standard canonical method of quantization. As is evident from the main body of our text, the non-vanishing canonical basic brackets are: $[\lambda, B] = i$, $[R, p_r] = i$, $\{C, \dot{\bar{C}}\} = +1$ and $\{\bar{C}, \dot{C}\} = -1$ and the rest of the brackets are zero. The trivial vanishing brackets are *eight* in number (i.e. $[R, R] = [p_r, p_r] = [\lambda, \lambda] = [B, B] = \{C, C\} = \{\bar{C}, \bar{C}\} = \{\dot{C}, \dot{C}\} = \{\dot{\bar{C}}, \dot{\bar{C}}\} = 0$) and the rest of the basic brackets, that are equal to zero, are:

$$\begin{aligned} [\lambda, C] &= [\lambda, \bar{C}] = [\lambda, \dot{C}] = [\lambda, \dot{\bar{C}}] = [\lambda, R] = [\lambda, p_r] = 0, \\ [B, C] &= [B, \bar{C}] = [B, \dot{C}] = [B, \dot{\bar{C}}] = [B, R] = [B, p_r] = 0, \\ \{C, \bar{C}\} &= \{C, \dot{C}\} = [R, C] = [p_r, C] = \{\dot{C}, \dot{\bar{C}}\} = [R, \dot{C}] = 0, \\ \{\bar{C}, \dot{\bar{C}}\} &= [R, \bar{C}] = [p_r, \bar{C}] = [R, \dot{\bar{C}}] = [p_r, \dot{\bar{C}}] = [p_r, \dot{C}] = 0. \end{aligned} \quad (58)$$

Thus, the total number of (non-)vanishing brackets at the level of variables and their conjugate momenta are *thirty six* in number. We shall express these in terms of the creation and annihilation operators of our present theory.

It is straightforward to note that the substitution of the mode expansions, in the above *thirty six* basic brackets, leads to the following sixty eight (68) vanishing basic (anti)commutators in terms of the creation and annihilation operators:

$$\begin{aligned} [s, s] &= [s^\dagger, s^\dagger] = [s, k] = [s^\dagger, k^\dagger] = [s, d] = [s, d^\dagger] = 0, \\ [s^\dagger, d] &= [s^\dagger, d^\dagger] = [s, l] = [s, l^\dagger] = [s^\dagger, l] = [s^\dagger, l^\dagger] = 0, \\ [s, c] &= [s, c^\dagger] = [s, \bar{c}] = [s, \bar{c}^\dagger] = [s^\dagger, c] = [s^\dagger, c^\dagger] = 0, \\ [s^\dagger, \bar{c}] &= [s^\dagger, \bar{c}^\dagger] = [k, k] = [k^\dagger, k^\dagger] = [k, d] = [k, d^\dagger] = 0, \\ [k^\dagger, d] &= [k^\dagger, d^\dagger] = [k, l] = [k, l^\dagger] = [k^\dagger, l] = [k^\dagger, l^\dagger] = 0, \\ [k, c] &= [k, c^\dagger] = [k, \bar{c}] = [k, \bar{c}^\dagger] = [k^\dagger, c] = [k^\dagger, c^\dagger] = 0, \\ [k^\dagger, \bar{c}] &= [k^\dagger, \bar{c}^\dagger] = [d, d] = [d^\dagger, d^\dagger] = [d, l] = [d^\dagger, l^\dagger] = 0, \\ [d, c] &= [d, c^\dagger] = [d, \bar{c}] = [d, \bar{c}^\dagger] = [d^\dagger, c] = [d^\dagger, c^\dagger] = 0, \\ [d^\dagger, \bar{c}] &= [d^\dagger, \bar{c}^\dagger] = [l, l] = [l^\dagger, l^\dagger] = [l, c] = [l, c^\dagger] = 0, \\ [l, \bar{c}] &= [l, \bar{c}^\dagger] = [l^\dagger, c] = [l^\dagger, c^\dagger] = [l^\dagger, \bar{c}] = [l^\dagger, \bar{c}^\dagger] = 0, \\ \{c, c\} &= \{c, c^\dagger\} = \{c^\dagger, c^\dagger\} = \{c, \bar{c}\} = \{\bar{c}, \bar{c}\} = \{\bar{c}, \bar{c}^\dagger\} = 0, \\ \{\bar{c}^\dagger, \bar{c}^\dagger\} &= \{c^\dagger, \bar{c}^\dagger\} = 0, \end{aligned} \quad (59)$$

along with the following non-vanishing canonical basic brackets

$$[d, l^\dagger] = [d^\dagger, l] = i, \quad [s, k^\dagger] = [s^\dagger, k] = i, \quad \{c, \bar{c}^\dagger\} = -i, \quad \{\bar{c}, c^\dagger\} = +i. \quad (60)$$

This exercise has been performed so that we can derive all these brackets from the symmetry principles and compare them in a precise manner. The salient features of the above basic

brackets are as follows. First, we note that there are only *four* independent brackets that are non-vanishing [cf. (60)]. We point out that the brackets $[d, l^\dagger] = i$ and $[d^\dagger, l] = i$ are Hermitian conjugate of each-other. Thus, only one of them is independent. Second, all the brackets in (59) are *not* independent (for instance, $[s^\dagger, c] = 0$ is equivalent to $[s, c^\dagger] = 0$ because these are Hermitian conjugate of each-other). Third, the basic brackets $\{c, \bar{c}^\dagger\} = -i$ and $\{\bar{c}, c^\dagger\} = +i$ are independent of each-other because the variables $C(t)$ and $\bar{C}(t)$ have been taken to be independent right from the beginning. Finally, if one of the bracket is calculated from the symmetry principles, its Hermitian conjugate would *also* be automatically true. This input has been taken into account in the main body of our text.

Appendix B: Logical Approximations and Mode Expansions

Here we discuss some of the details of our approximation as well as solution of the equations of motion: $B = -(\dot{\lambda} - p_r)$, $B = d/dt(r - a)$, $\dot{B} = -(r - a)$, $\dot{p}_r + \lambda = 0$ which emerge from the approximated Lagrangian $L_B^{(0)} = \dot{r} p_r - \lambda(r - a) + B(\dot{\lambda} - p_r) + B^2/2 - i\dot{\bar{C}}\dot{C} + i\bar{C}\dot{C}$. In the latter, one term (i.e. $(r^2\dot{\theta}^2)/2$) has been ignored for a rigid rotor with a constant angular velocity $\dot{\theta}$ (i.e. $\dot{\theta} = \text{constant}$). The above EL equations of motion imply that $\ddot{B} + B = 0$, $\ddot{R} + R = 0$ and $\frac{d^2}{dt^2}(\dot{\lambda} - p_r) + (\dot{\lambda} - p_r) = 0$ where $R = (r - a)$. Using the equation of motion $\dot{p}_r + \lambda = 0$, one can clearly observe that the following is true, namely;

$$\frac{d^2}{dt^2}(\dot{\lambda} - p_r) + (\dot{\lambda} - p_r) = 0 \implies \frac{d^2}{dt^2}(\ddot{p}_r + p_r) + (\ddot{p}_r + p_r) = 0, \quad (61)$$

which is a common feature of an equation of motion for a harmonic oscillator in terms of the variable $p_r(t)$ (i.e. $\ddot{p}_r + \omega^2 p_r = 0$ with frequency $\omega = 1$). It is evident that $\frac{d^2}{dt^2}(\ddot{p}_r + p_r) + (\ddot{p}_r + p_r) = 0$ would be satisfied if we set $\ddot{p}_r + p_r = 0$. It may be worthwhile to mention that all equations like $\frac{d^{2n}}{dt^{2n}}(\ddot{p}_r + p_r) + \frac{d^{(2n-2)}}{dt^{(2n-2)}}(\ddot{p}_r + p_r) = 0$ ($n = 1, 2, 3, \dots$) would be always satisfied for the EOM connected to the harmonic oscillator $\ddot{p}_r + p_r = 0$ with frequency $\omega = 1$. The solution of (61) is the one which is given in the mode expansion (18). The equation of motion $\dot{p}_r + \lambda = 0$ implies that one of the interesting solutions of this equation of motion that could be satisfied by λ would be $\ddot{\lambda} + \lambda = 0$ whose mode expansion is given in (18).

We would like to observe that the normal mode expansion (18) have been taken in a uniform manner because $\ddot{C} + C = 0$, $\ddot{\bar{C}} + \bar{C} = 0$, $\ddot{R} + R = 0$, $\ddot{B} + B = 0$ emerge automatically but $\ddot{p}_r + p_r = 0$ and $\ddot{\lambda} + \lambda = 0$ come out due to some approximations. It can be checked that if we take the constraint equations: $R \approx 0$ and $d/dt(R) \approx 0$, the equations of motion $\ddot{\lambda} + \lambda = 0$ and $\ddot{p}_r + p_r = 0$ emerge very naturally from our theory. We lay emphasis on the fact that the equations of motion (i.e. $\ddot{p}_r + p_r = 0$ and $\ddot{\lambda} + \lambda = 0$) for $p_r(t)$ and $\lambda(t)$ are due to specific approximations but these are the ones which are interesting for our purposes.

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